Bigravity in Kuchař's Hamiltonian formalism. 1. The general case

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Abstract

The Hamiltonian formalism of bigravity and bimetric theories is constructed for the general form of potential between two metrics. It is natural to study the role played by the lapse and shift functions in theories with two metrics on the base of Kuchař's approach because then they do not depend on the choice of space-time coordinate frame. The algebra of first class constraints is derived in Dirac brackets constructed from the second class constraints. It is the celebrated algebra of the hypersurface deformations. Fixing one of the metrics we obtain a bimetric theory without any first class constrains. Then we can use the symmetries of the background metric to construct conserved quantities looking ultralocally when written through the potential. The special case of potential providing the less number for degrees of freedom will be treated in the companion paper.

1 Introduction

The idea to use more than one dynamical metric for the description of the real Universe is not rather new. This theory is usually called multi-gravity. There is a hope that such a modification of the gravitational theory may allow to solve the dark energy problem. We restrict ourselves to the two metrics case, and will call such a model as bigravity, following pioneers [1]. Suppose each metric interacts only with its own sort of matter, but these metrics can interact with each other by means of some potential, i.e. a scalar density algebraically constructed of the two metrics. If we fix one metric by giving

it the absolute meaning, we move from bigravity to bimetric theory, as since N. Rosen's [2] times the metric theories of gravity on the fixed background are called. In particular, massive gravity theories [3] applied for explaining the dark energy phenomena in the Universe are in this category.

We plan to consider the Hamiltonian formalism of bigravity in two papers dividing our presentation in two parts which we call the general case and the special case. In general case the concrete form of interaction potential is not given, we only say that it is constructed algebraically (ultralocally, i.e. without derivatives), it is a scalar density and does not contain any fields of matter. We call the special case the model with a potential giving one degree of freedom less than the general case. The form of such a potential had been proposed recently [4]. Unfortunately straightforward calculations with this potential in Hamiltonian formalism are difficult because of its matrix square root form, and so the analysis given already in a publications [5] does not look transparent and decisive. We are intended to derive the requirements on the potential which are necessary for realizing the program proclaimed in papers [4], starting from a potential of the general form. In contrast to the previous articles on this subject we exploit not ADM, but Kuchař's approach which preserves the freedom of spacetime coordinates choice.

We denote as $f_{\mu\nu}$ and call as the first metric the metric that will be fixed (i.e. become a background one) after a shift from bigravity to bimetric theory, and the second metric $g_{\mu\nu}$ which will always stay dynamical. Greek indices will run from 0 to 3, Latin – from 1 to 3, signature is $(-,+,+,+)$.

Consider two copies of General Relativity Lagrangian, each copy supplied with its own sort of matter as a source (interaction with matter is supposed minimal),

$$\mathcal{L}^{(f)} = \frac{1}{16\pi G^{(f)}}\sqrt{-f}f^{\mu\nu}R^{(f)}_{\mu\nu} + \mathcal{L}^{(f)}_M(\psi^A, f_{\mu\nu}), \tag{1}$$

$$\mathcal{L}^{(g)} = \frac{1}{16\pi G^{(g)}}\sqrt{-g}g^{\mu\nu}R^{(g)}_{\mu\nu} + \mathcal{L}^{(g)}_M(\phi^A, g_{\mu\nu}), \tag{2}$$

where $f_{\mu\nu}$ and $g_{\mu\nu}$ are the first and the second spacetime metrics correspondingly, f and g are their determinants, $R^{(f)}_{\mu\nu}$ and $R^{(g)}_{\mu\nu}$ are Ricci tensors, $G^{(f)}$ and $G^{(g)}$ are gravitational constants, $\mathcal{L}^{(g)}_M$ and $\mathcal{L}^{(f)}_M$ are Lagrangians of the first and the second matter, A denotes abstract indicies for matter fields, and construct of them a new Lagrangian by adding a potential

$$\sqrt{-f}U(f_{\mu\nu}, g_{\mu\nu}), \tag{3}$$

therefore,

$$\mathcal{L} = \mathcal{L}^{(f)} + \mathcal{L}^{(g)} + \sqrt{-f}U(f_{\mu\nu}, g_{\mu\nu}). \tag{4}$$

It is clear that a choice of metric determinant which is present explicitly is not of importance because function $U(f_{\mu\nu}, g_{\mu\nu})$ may arbitrarily depend on the ratio of two determinants.

2 Kuchař's Hamiltonian formalism in General Relativity

Let the action of General Relativity (GR) is written as follows

$$S = \int \mathcal{L} d^4 X, \tag{5}$$

where X^α are spacetime coordinates, and the form of Lagrangian is the same as in formulas (1), (2).

In the construction of Hamiltonian formalism it is necessary to separate explicitly the time coordinate from the spatial ones. The state should be determined by values of the gravitational and matter fields given at all points of space at one definite moment of time. So, the state is to be given on a spacelike hypersurface embedded in spacetime. The flow of time corresponds to the motion of this hypersurface through spacetime or to the continuous transform from one hypersurface to another, so we need one-parametrical family of spacelike hypersurfaces. The role of time coordinate t is to be played by a parameter which continuously and monotonically numerates hypersurfaces. It is suitable to introduce spatial coordinates x^i on the initial hypersurface and then continue them to all hypersurfaces in such a way that lines going through the points with the same values of coordinates can be treated as observer world lines, i.e. they should be timelike.

In ADM approach [6], which is the most popular, the choice of spacelike hypersurfaces family is determined by the choice of spacetime coordinate frame:

$$t = X^0, \quad x^i = X^i, \quad \gamma_{ij} = g_{ij}, \quad N = (-g^{00})^{-1/2}, \quad N_i = g_{0i}. \tag{6}$$

There is another approach proposed by Kuchař [7] where two coordinate systems are in action, one is an arbitrary spacetime coordinate system X^α,

the other (t, x^i) is defined by embedding variables

$$X^\alpha = e^\alpha(x^i, t), \tag{7}$$

where time is a parameter monotonically numerating hypersurfaces and other three coordinates numerate points on a hypersurface. We will follow this fully covariant Kuchař's method. Then fields

$$e^\alpha_i \equiv \frac{\partial e^\alpha}{\partial x^i} \tag{8}$$

will simultaneously be vectors in spacetime and covectors in space. The metric induced on a hypersurface is given as follows

$$\gamma_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j. \tag{9}$$

As usual, inverse matrices to metrics $g_{\mu\nu}$ and γ_{ij} are denoted as $g^{\mu\nu}$ and γ^{ij}. Moving indices (both Greek and Latin) up and down is provided by means of them:

$$\bar{e}_{\alpha i} = g_{\alpha\beta} e^\beta_i, \quad \bar{e}^i_\alpha = g_{\alpha\beta} e^\beta_j \gamma^{ij}. \tag{10}$$

We will use here a bar to make a distinction between the quantities defined by means of metric and the quantities e^α_i which are independent of the metric. In the following, when we will work with two metrics, the bar will stay connected with $g_{\mu\nu}$.

Next we introduce normal 1-form

$$n_\alpha e^\alpha_i = 0, \tag{11}$$

which can be converted with the help of metric tensor to a normalized vector:

$$\bar{n}^\alpha = g^{\alpha\beta} \bar{n}_\beta, \qquad g^{\mu\nu} \bar{n}_\mu \bar{n}_\nu = -1. \tag{12}$$

We can use basis $(\bar{n}^\alpha, e^\alpha_i)$ to decompose every vector or tensor in spacetime, for example, Ricci tensor (with account for its symmetric nature):

$$R^{\mu\nu} = R^{\perp\perp} \bar{n}^\mu \bar{n}^\nu + R^{i\perp}(e^\mu_i \bar{n}^\nu + \bar{n}^\mu e^\nu_i) + R^{ij} e^\mu_i e^\nu_j, \tag{13}$$

here the components are calculated as follows:

$$R^{\perp\perp} = R^{\mu\nu} \bar{n}_\mu \bar{n}_\nu, \qquad R^{i\perp} = -R^{\mu\nu} \bar{n}_\mu \bar{e}_{\nu i}, \qquad R^{ij} = R^{\mu\nu} \bar{e}_{\mu i} \bar{e}_{\nu j}. \tag{14}$$

With the known technique [7, 8], based on expression of the Riemann tensor projections as the corresponding projections of covariant derivatives commutators, it is possible to write GR Lagrangian density containing the single spacetime metric $g_{\mu\nu}$ in the following form

$$\mathcal{L}^g = \frac{\bar{N}\sqrt{\gamma}}{\kappa^{(g)}}\left(R^{(\gamma)} - \bar{K}^2 + Sp\bar{K}^2\right) + \mathcal{L}_M(\phi^A, \gamma_{ij}, \bar{N}, \bar{N}^i), \tag{15}$$

where boundary terms, i.e. total time derivatives and spatial divergences are ignored, as we do not discuss boundary conditions. Here $\kappa^{(g)} = 1/16\pi G^{(g)}$, $R^{(\gamma)}$ is the scalar curvature of the induced metric, $\gamma = \det||\gamma_{ij}||$, lapse and shift functions $\bar{N}, \bar{N}^i$ are components of the decomposition of time vector field over the basis constructed with metric $g_{\mu\nu}$

$$N^\alpha \equiv \frac{\partial X^\alpha}{\partial t} = \bar{N}\bar{n}^\alpha + \bar{N}^i e_i^\alpha. \tag{16}$$

The second fundamental form $\bar{K}_{ij}$ arising in the Lagrangian density can be expressed through the already introduced variables by formula:

$$\bar{K}_{ij} = \frac{1}{2\bar{N}}\left(\bar{N}_{i|j} + \bar{N}_{j|i} - \dot{\gamma}_{ij}\right), \tag{17}$$

here $\bar{K} = \gamma^{ij}\bar{K}_{ij}$, $\mathrm{Sp}\bar{K}^2 = \bar{K}_{ij}\bar{K}^{ij}$, vertical bar denotes a covariant derivative defined by induced metric. The momenta conjugate to variables γ_{ij} and ϕ^A are determined as follows:

$$\pi^{ij} = \frac{\partial\mathcal{L}^{(g)}}{\partial\dot{\gamma}_{ij}} = -\frac{\sqrt{\gamma}}{\kappa^{(g)}}(\bar{K}^{ij} - \gamma^{ij}\bar{K}), \quad \pi_A = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^A}, \tag{18}$$

and the momenta conjugate to $\bar{N}$ and $\bar{N}^i$ are zero, as expression (15) does not contain velocities $\dot{\bar{N}}$, $\dot{\bar{N}}^i$

$$\pi_{\bar{N}} = 0, \qquad \pi_{\bar{N}^i} = 0, \tag{19}$$

so, these equations are primary constraints. The velocities for gravitational variables γ_{ij} can be expressed by means of momenta

$$\dot{\gamma_{ij}} = \bar{N}_{i|j} + \bar{N}_{j|i} + \frac{2\kappa^{(g)}\bar{N}}{\sqrt{\gamma}}(\pi_{ij} - \gamma_{ij}\frac{\pi}{2}). \tag{20}$$

We believe that it is possible to do the same for the matter variables (in other case we can analyse new constraints arising there), and after performing the Legendre transform

$$\mathrm{H}_{\mathrm{canonical}} = \int d^3x \left(\pi^{ij}\dot{\gamma}_{ij} + \pi_A \dot{\phi}^A - \mathcal{L} \right), \tag{21}$$

to obtain the canonical Hamiltonian (up to surface terms)

$$\mathrm{H}_{\mathrm{canonical}} = \int d^3x \left(\bar{N}\bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i \right). \tag{22}$$

Poisson brackets for functionals of canonical variables are as follows

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta \bar{N}} \frac{\delta G}{\delta \pi_{\bar{N}}} + \frac{\delta F}{\delta \bar{N}^i} \frac{\delta G}{\delta \pi_{\bar{N}^i}} + \frac{\delta F}{\delta \gamma_{ij}} \frac{\delta G}{\delta \pi^{ij}} + \frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \pi_A} - (F \leftrightarrow G) \right). \tag{23}$$

According to Dirac method [9], one should further go to the extended Hamiltonian by adding the primary constraints with arbitrary Lagrangian multipliers

$$\mathrm{H}_{\mathrm{extended}} = \int d^3x \left(\bar{N}\bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i + \lambda \pi_{\bar{N}} + \lambda^i \pi_{\bar{N}^i} \right). \tag{24}$$

Then expressions

$$\bar{\mathcal{H}} = -\frac{1}{\sqrt{\gamma}} \left(\frac{1}{\kappa^{(g)}} \gamma R^{(\gamma)} + \kappa^{(g)} \left(\frac{\pi^2}{2} - \mathrm{Sp}\pi^2 \right) \right) + \bar{\mathcal{H}}_M \tag{25}$$

and

$$\bar{\mathcal{H}}_i = -2\pi^j_{i|j} + \bar{\mathcal{H}}_{iM} \tag{26}$$

become secondary constraints, as their equality to zero is necessary for primary constraints staying zero in evolution. Here $\pi = \gamma_{ij}\pi^{ij}$, $\mathrm{Sp}\pi^2 = \pi^{ij}\pi_{ij}$, $\bar{\mathcal{H}}_M, \bar{\mathcal{H}}_{iM}$ is the matter contribution.

The standard procedure requires to remove four pairs of canonical variables $\bar{N}, \bar{N}^i$, $\pi_{\bar{N}}, \pi_{\bar{N}^i}$ and of primary constraints (19) by putting on four gauge conditions which really change only the meaning of four letters, i.e. replace canonical variables with functions (we do not like to introduce new notations and we make this change only here)

$$\bar{N} - \bar{\mathcal{N}}(x) = 0, \qquad \bar{N}^i - \bar{\mathcal{N}}^i(x) = 0. \tag{27}$$

These gauges have nonzero Poisson brackets with primary constraints (19) and allow to treat all of them together as eight second class constraints, this leads to Dirac brackets:

$$\{F,G\}_D = \int d^3x \left(\frac{\delta F}{\delta \gamma_{ij}} \frac{\delta G}{\delta \pi^{ij}} + \frac{\delta F}{\delta \phi_A} \frac{\delta G}{\delta \pi^A} - (F \leftrightarrow G) \right). \tag{28}$$

The number of gravitational degrees of freedom is now determined by a simple calculation: take the number of induced metric γ_{ij} independent components and subtract the number of first class constraints: $6-4=2$, now Hamiltonian takes a following form

$$\mathrm{H}_{\text{partially reduced}} = \int d^3x \left(\bar{\mathcal{N}} \bar{\mathcal{H}} + \bar{\mathcal{N}}^i \bar{\mathcal{H}}_i \right). \tag{29}$$

Avoiding cumbersome terminology and notations we as usual will call Dirac brackets (28) as Poisson brackets, the partially reduced Hamiltonian (29) as Hamiltonian, $\bar{N} = \bar{\mathcal{N}}$, $\bar{N}^i = \bar{\mathcal{N}}^i$ as Lagrangian multipliers standing before constraints (25), (26). These constraints are first class as their algebra is as follows

$$\{\bar{\mathcal{H}}(x), \bar{\mathcal{H}}(y)\} = (\eta^{ik}(x)\bar{\mathcal{H}}_k(x) + \eta^{ik}(y)\bar{\mathcal{H}}_k(y))\delta_{,i}(x,y), \tag{30}$$

$$\{\bar{\mathcal{H}}_i(x), \bar{\mathcal{H}}_k(y)\} = \bar{\mathcal{H}}_i(y)\delta_{,k}(x,y) + \bar{\mathcal{H}}_k(x)\delta_{,i}(x,y), \tag{31}$$

$$\{\bar{\mathcal{H}}_i(x), \bar{\mathcal{H}}(y)\} = \bar{\mathcal{H}}(x)\delta_{,i}(x,y), \tag{32}$$

reflecting a freedom of hypersurface deformations in Riemannian space.

3 Hamiltonian approach in bigravity

Having two spacetime metrics we get two induced metrics on a spatial hypersurface:

$$\gamma_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j, \qquad \eta_{ij} = f_{\mu\nu} e^\mu_i e^\nu_j, \tag{33}$$

and also two different unit normal vectors that will be denoted as n^α $\bar{n}^\alpha$:

$$n_\alpha e^\alpha_i = 0 = \bar{n}_\alpha e^\alpha_i, \quad f^{\alpha\beta} n_\alpha n_\beta = -1, \quad g^{\alpha\beta} \bar{n}_\alpha \bar{n}_\beta = -1, \tag{34}$$

and two bases (n^α, e^α_i), $(\bar{n}^\alpha, e^\alpha_i)$. Moving indices up and down is provided by the corresponding metric tensors $f_{\mu\nu}, \eta_{ij}$ and $g_{\mu\nu}, \gamma_{ij}$. We will decompose any spacetime vectors and tensors over basis (n^α, e^α_i), constructed with the

help of $f_{\mu\nu}$ which will be called the first metric. For example, we decompose the second metric tensor (with account for its symmetry) as follows

$$g^{\mu\nu} = g^{\perp\perp} n^\mu n^\nu + g^{\perp i}(e_i^\mu n^\nu + n^\mu e_i^\nu) + g^{ij} e_i^\mu e_j^\nu, \tag{35}$$

whereas,

$$f^{\mu\nu} = -n^\mu n^\nu + e_i^\mu e_j^\nu \eta^{ij}, \tag{36}$$

the components are calculated by formulas:

$$g^{\perp\perp} = g^{\mu\nu} n_\mu n_\nu, \quad g^{\perp i} = -g^{\mu\nu} n_\mu e_\nu^i, \quad g^{ij} = g^{\mu\nu} e_\mu^i e_\nu^j = \gamma^{ij} + \frac{g^{\perp i} g^{\perp j}}{g^{\perp\perp}}. \tag{37}$$

We write each metric contribution to bigravity Lagrangian (4) in the similar way to equation (15). The potential, in its turn, depends on the two space-time metrics, therefore it is necessary to brake their symmetry and take one of the metrics for construction of the basis in order to transform the potential to $(3+1)$-form. Previously we already have decided to call $f_{\mu\nu}$ the first metric and to use the basis constructed with its help. Then besides lapse N and shift N^i

$$N^\alpha = N n^\alpha + N^i e_i^\alpha = \bar{N} \bar{n}^\alpha + \bar{N}^i e_i^\alpha \tag{38}$$

and induced spatial metrics η_{ij}, γ_{ij}, the potential depends on other four components of metric $g^{\mu\nu}$ decomposition over basis (n^α, e_i^α), let us accentuate that in contrast to N, quantities $g^{\perp\perp}, g^{\perp i}$ enter the potential in a nonlinear way. Relations between the two bases

$$\bar{n}^\alpha = \sqrt{-g^{\perp\perp}}\, n^\alpha - \frac{g^{\perp i}}{\sqrt{-g^{\perp\perp}}} e_i^\alpha \tag{39}$$

allow us to express lapse and shift of metric $g_{\mu\nu}$, i.e. $\bar{N}$, $\bar{N}^i$ through N, N^i and normal projections $g^{\perp\perp}, g^{\perp i}$:

$$\bar{N} = \frac{N}{\sqrt{-g^{\perp\perp}}}, \qquad \bar{N}^i = N^i - \frac{g^{\perp i}}{g^{\perp\perp}} N. \tag{40}$$

It is suitable for the following to introduce new variables

$$u = \frac{1}{\sqrt{-g^{\perp\perp}}}, \qquad u^i = -\frac{g^{\perp i}}{g^{\perp\perp}}, \tag{41}$$

having the simple geometric meaning: u is an inverse of a norm (calculated in the second metric) of vector n^α, constructed as a unit normal (in the first metric) to the hypersurface, and u^i are three projections (calculated in the second metric) of coordinate basis vectors onto this unit normal

$$u = \frac{1}{\sqrt{|g^{\mu\nu} n_\mu n_\nu|}}, \qquad u^i = \frac{g^{\mu\nu} n_\mu e^i_\nu}{\sqrt{|g^{\mu\nu} n_\mu n_\nu|}}. \tag{42}$$

Then equations (40) takes a form

$$\bar{N} = uN, \qquad \bar{N}^i = N^i + u^i N. \tag{43}$$

We consider as dynamic variables two sets of matter fields ϕ_A, ψ_A, two induced metrics on the hypersurface η_{ij}, γ_{ij}, components of the time vector in the chosen basis (i.e. lapse and shift) N, N^i and four additional variables u, u^i, taken intead of the second metric projections $g^{\perp\perp}, g^{\perp i}$. Momenta are defined in the usual way

$$\Pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\eta}_{ij}}, \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}}, \quad \Pi_A = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^A}, \quad \pi_A = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A}. \tag{44}$$

The two procedures of defining momenta and Legendre transform are realized in parallel and independent way for both terms $\mathcal{L}^f$ and $\mathcal{L}^g$. Primary constraints arise because velocities of eight variables N, N^i, u, u^i are absent in the Lagrangian density, these constraints have a following form

$$\pi_N = 0, \quad \pi_{N^i} = 0, \tag{45}$$

$$\pi_u = 0, \quad \pi_{u^i} = 0. \tag{46}$$

Certainly, some additional primary constraints can appear due to gauge invariance which matter fields may have, but we will not discuss this here. The canonical Hamiltonian is derived by the Legendre transform

$$\mathrm{H}_{\text{canonical}} = \int d^3x \left(\Pi^{ij} \dot{\eta}_{ij} + \pi^{ij} \dot{\gamma}_{ij} + \pi_{\psi^A} \dot{\psi}^A + \pi_{\phi^A} \dot{\phi}^A - \mathcal{L} \right) \tag{47}$$

and expression of velocities through the corresponding momenta, this provides us with the two similar looking terms without common variables plus the potential:

$$\mathrm{H}_{\text{canonical}} = \int d^3x \left(N\mathcal{H} + N^i \mathcal{H}_i + \bar{N}\bar{\mathcal{H}} + \bar{N}^i \bar{\mathcal{H}}_i + N\sqrt{\eta} U \right), \tag{48}$$

or in other way,

$$\mathrm{H}_{\text{canonical}} = \int d^3x \left(N \left(\mathcal{H} + u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \sqrt{\eta}U \right) + N^i \left(\mathcal{H}_i + \bar{\mathcal{H}}_i \right) \right). \tag{49}$$

The canonical Poisson brackets have a standard appearance corresponding to the set of conjugate variables (η_{ij}, Π^{ij}), (γ_{ij}, π^{ij}), (ψ^A, Π_A), (ϕ^A, π_A), (N, π_N), (N^i, π_{N^i}), (u, π_u), (u^i, π_{u^i}). The requirement of conservation of the primary constraints (45), (46) in Hamiltonian evolution forces us to put their Poisson brackets with Hamiltonian to zero

$$\{\pi_N, \mathrm{H}\} = 0, \quad \{\pi_{N^i}, \mathrm{H}\} = 0, \quad \{\pi_u, \mathrm{H}\} = 0, \quad \{\pi_{u^i}, \mathrm{H}\} = 0, \tag{50}$$

this leads to the secondary constraints:

$$\mathcal{R} \equiv \mathcal{H} + u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \sqrt{\eta}U = 0, \tag{51}$$

$$\mathcal{R}_i \equiv \mathcal{H}_i + \bar{\mathcal{H}}_i = 0, \tag{52}$$

$$\bar{\mathcal{H}} + \frac{\partial \tilde{U}}{\partial u} = 0, \qquad \bar{\mathcal{H}}_i + \frac{\partial \tilde{U}}{\partial u^i} = 0, \tag{53}$$

where $\tilde{U} = \sqrt{\eta}U$. In the general case which is under discussion in this paper the four last constraints together with the four last primary constraints occurs second class [10] and can be excluded from Hamiltonian after introduction of Dirac brackets (see Appendix B). The special case will be treated in the companion article. For other secondary constraints the algebra of hypersurface deformations should be fulfilled:

$$\{\mathcal{R}(x), \mathcal{R}(y)\}_D = \left[\eta^{ik}(x)\mathcal{R}_k(x) + \eta^{ik}(y)\mathcal{R}_k(y)\right] \delta_{,i}(x, y), \tag{54}$$

$$\{\mathcal{R}_i(x), \mathcal{R}_k(y)\}_D = \mathcal{R}_i(y)\delta_{,k}(x, y) + \mathcal{R}_k(x)\delta_{,i}(x, y), \tag{55}$$

$$\{\mathcal{R}_i(x), \mathcal{R}(y)\}_D = \mathcal{R}(x)\delta_{,i}(x, y). \tag{56}$$

Calculating l.h.s. first as Poisson brackets, then applying second class constraints (53), at last comparing results with r.hs., we derive conditions on the potential given as function of variables $\eta_{ij}, \gamma_{ij}, u, u^i$ (details see in Appendix A):

$$2\eta_{jk}\frac{\partial \tilde{U}}{\partial \eta_{ij}} + 2\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{ij}} - u^i\frac{\partial \tilde{U}}{\partial u^k} = \delta^i_k\tilde{U}, \tag{57}$$

$$2u^j\gamma_{jk}\frac{\partial \tilde{U}}{\partial \gamma_{k\ell}} - u^\ell u\frac{\partial \tilde{U}}{\partial u} + \left(\eta^{k\ell} - u^2\gamma^{k\ell} - u^k u^\ell\right)\frac{\partial \tilde{U}}{\partial u^k} = 0. \tag{58}$$

These conditions preserve their form under a change of variables corresponding to the interchange of bases and roles of the two metrics:

$$u \to \frac{1}{u}, \quad u^i \to -\frac{u^i}{u}, \quad \eta_{ij} \leftrightarrow \gamma_{ij}, \quad \tilde{U} \to u\tilde{U}. \tag{59}$$

The simplest method to derive Hamiltonian equations is to use canonical Hamiltonian and Poisson brackets. Then they are looking similar to GR Hamiltonian equations (here nabla denotes a covariant derivative defined by η_{ij}):

$$\begin{aligned}
\dot{\eta}_{ij} &= \{\eta_{ij}, \mathrm{H}^f\} = \nabla_j N_i + \nabla_i N_j + \kappa^{(f)} \frac{2N}{\sqrt{\eta}} (\Pi_{ij} - \eta_{ij}\frac{\Pi}{2}), & (60)\\
\dot{\gamma}_{ij} &= \{\gamma_{ij}, \mathrm{H}^g\} = \bar{N}_{i|j} + \bar{N}_{j|i} + \kappa^{(g)} \frac{2\bar{N}}{\sqrt{\gamma}} (\pi_{ij} - \gamma_{ij}\frac{\pi}{2}), & (61)\\
\dot{\Pi}^{ij} &= \{\Pi^{ij}, \mathrm{H}^f\} - N\frac{\partial \tilde{U}}{\partial \eta_{ij}}, & (62)\\
\dot{\pi}^{ij} &= \{\pi^{ij}, \mathrm{H}^g\} - N\frac{\partial \tilde{U}}{\partial \gamma_{ij}}, & (63)\\
\dot{\psi}^A &= \{\psi^A, H^{(f)}\}, & (64)\\
\cdots &= \cdots & (65)
\end{aligned}$$

but one should remember that if N, N^i are arbitrary Lagrangian multipliers (before putting on gauge conditions), u, u^i after solving second class constraints become functions of $\bar{\mathcal{H}}, \bar{\mathcal{H}}_i$, which in their turn are expressed by equations (25), (26) through canonical variables. Certainly, these Hamiltonian equations can be derived by means of Dirac brackets from Hamiltonian after applying second class constraints to it:

$$\mathrm{H} = \int d^3x \left(N \left(\mathcal{H} + \tilde{U} - u\frac{\partial \tilde{U}}{\partial u} - u^i \frac{\partial \tilde{U}}{\partial u^i} \right) + N^i \left(\mathcal{H}_i - \frac{\partial \tilde{U}}{\partial u^i} \right) \right). \tag{66}$$

When we are given embedding variables $X^\alpha = e^\alpha(t, x^i)$, we can express both spacetime metrics in coordinates X^α, if we first estimate the unit normal vector

$$n^\alpha = \frac{1}{N} \left(\frac{\partial e^\alpha}{\partial t} - N^i e^\alpha_i \right), \tag{67}$$

and then apply formulas (35), (36), (37) for two metrics decompositions in this basis.

4 Hamiltonian approach in bimetric theories

Consider now the situation when only the second metric is dynamical and the first one is background, i.e. it is a fixed solution of GR equations. Traditionally [2] such theories are called bimetric. Recently they attracted a lot of attention and some valuable reviews have appeared [11, 12, 13, 14].

If metric tensor $f_{\mu\nu}$ and embedding variables $e^\alpha(t,x^k)$ are given, then we already know functions $N(t,x^k)$, $N^i(t,x^k)$, $\eta_{ij}(t,x^k)$. We can take expression (49), derived above for bigravity, as a Hamiltonian, treating N, N^i not as variables, but as given parameters, and taking into account that $\mathcal{H}, \mathcal{H}_i$ are now zero,

$$H = \int d^3x \left(N \left(u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \sqrt{\eta}U \right) + N^i\bar{\mathcal{H}}_i \right). \tag{68}$$

Then primary constraints are only equations (46), and they lead to secondary constraints (53). In the general case (remember, that special case will be treated in another paper) all these constraints are second class, and Dirac brackets constructed from them will coincide with Poisson brackets for the functionals which depend only on the independent variables γ_{ij}, π^{ij}, ϕ^A, π_A:

$$\{F,G\}_D = \int d^3x \left(\frac{\delta F}{\delta\gamma_{ij}}\frac{\delta G}{\delta\pi^{ij}} + \frac{\delta F}{\delta\phi_A}\frac{\delta G}{\delta\pi^A} - (F \leftrightarrow G) \right). \tag{69}$$

For example, Hamiltonian fits in this category after excluding variables u, π_u, u^i, π_{u^i} which can be done with the hekp of equations (46), (53). Just opposite, we can express Hamiltonian by means of the potential $\tilde{U}$ and its derivatives with respect to variables u, u^i:

$$\mathrm{H} = \int d^3x \left(N \left(\tilde{U} - u\frac{\partial\tilde{U}}{\partial u} - u^i\frac{\partial\tilde{U}}{\partial u^i} \right) - N^i\frac{\partial\tilde{U}}{\partial u^i} \right), \tag{70}$$

in this case more complicated formulas for Dirac brackets which take into account variables u, u^i are to be used, see Appendix B.

Hamiltonian of bimetric theory is not a linear combination of first class constraints, in contrast to bigravity and GR Hamiltonians, so we can use it for construction of conserved quantities. For example, if the background metric is flat, then the Hamiltonian density may be interpreted as a density of energy, momentum or angular momentum, and all these quantities are ultralocal functions of the corresponding lapse and shift N, N^i, two induced metrics η_{ij}, γ_{ij} and u, u^i.

5 Conclusion

In this article we have studied the theory of bigravity with a potential of the general form and built the Hamiltonian formalism, avoiding unnecessary noncovariance in defining the lapse and shift functions, which we consider as elderly notation in the powerful ADM approach. The potential in this work should satisfy only two essential conditions: matrix $\mathbf{L}$ (94) is to be nondegenerate and the first class constraints are to fulfill the algebra of hypersurface deformations (necessary and sufficient conditions for this are equations (57), (58)). The special potential choice proposed by de Rham-Gabadadze,Tolley [4] will be treated in another article.

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A First class constraints algebra in bigravity

In two metric theory, as in GR, there are four first class constraints and their algebra is the algebra of hypersurface deformations. In computation of Poisson brackets between constraints $\mathcal{R}$, $\mathcal{R}_i$ we treat u, u^i not as canonical variables, but as functions, because their conjugate momenta π_u, π_{u^i} do not appear in the computation. Then potential $\tilde{U}$ in these calculations has nonzero Poisson brackets with gravitational momenta Π^{ij}, π^{ij} only, giving as results its derivatives with respect to induced metrics $\partial\tilde{U}/\partial\eta_{ij}$ and $\partial\tilde{U}/\partial\gamma_{ij}$. First, consider equation (55), generators $\mathcal{H}_i$ and $\bar{\mathcal{H}}_i$ are mutually commuting, and each of them separately satisfies equation (31), therefore, equation (55) is also fulfilled. L.h.s. of equation (54) can be given as follows

$$\begin{aligned}
\{\mathcal{R}(x), \mathcal{R}(y)\} &= \{\mathcal{H}(x), \mathcal{H}(y)\} + && (71)\\
&+ \{u\bar{\mathcal{H}}(x), u\bar{\mathcal{H}}(y)\} + \{u^i\bar{\mathcal{H}}_i(x), u^j\bar{\mathcal{H}}_j(y)\} + && (72)\\
&+ \{u\bar{\mathcal{H}}(x), u^i\bar{\mathcal{H}}_i(y)\} + \{u^i\bar{\mathcal{H}}_i(x), u\bar{\mathcal{H}}(y)\} + && (73)\\
&+ \{\mathcal{H}(x), \tilde{U}(y)\} + \{\tilde{U}(x), \mathcal{H}(y)\} + && (74)\\
&+ \{u\bar{\mathcal{H}}(x), \tilde{U}(y)\} + \{\tilde{U}(x), u\bar{\mathcal{H}}(y)\} + && (75)\\
&+ \{u^i\bar{\mathcal{H}}_i(x), \tilde{U}(y)\} + \{\tilde{U}(x), u^i\bar{\mathcal{H}}_i(y)\}. && (76)
\end{aligned}$$

Lines from 1 to 3 can be easily computed by using formulas (30), (31), (32):

$$\begin{aligned}
&= \Big(\eta^{ik}(x)\mathcal{H}_k(x) + \eta^{ik}(y)\mathcal{H}_k(y)\Big)\,\delta_{,i}(x,y) + && (77)\\
&+ u(x)u(y)\Big(\gamma^{ik}(x)\bar{\mathcal{H}}_k(x) + \gamma^{ik}(y)\bar{\mathcal{H}}_k(y)\Big)\,\delta_{,i}(x,y) + && (78)\\
&+ u^i(x)u^k(y)\Big(\bar{\mathcal{H}}_i(y)\delta_{,k}(x,y) + \bar{\mathcal{H}}_k(x)\Big)\,\delta_{,i}(x,y)) - && (79)\\
&- u^i(x)u(y)\bar{\mathcal{H}}(x)\delta_{,i}(x,y) + u^i(y)u(x)\bar{\mathcal{H}}(y)\delta_{,i}(y,x), && (80)
\end{aligned}$$

line 4 and line 5 give zero results, as they are antisymmetric in x, y whereas each one contain δ-function. Last line gives a contribution

$$2u^j\gamma_{jk}\frac{\partial\tilde{U}}{\partial\gamma_{ik}}(x)\delta_{,i}(x,y) - 2u^j\gamma_{jk}\frac{\partial\tilde{U}}{\partial\gamma_{ik}}(y)\delta_{,i}(y,x). \qquad (81)$$

Taking into account relations

$$f(y)\delta_{,i}(x,y) = f(x)\delta_{,i}(x,y) + f_{,i}\delta(x,y), \qquad \delta_{,i}(x,y) = -\delta_{,i}(y,x), \qquad (82)$$

we can transform the result to the following form:

$$\{\mathcal{R}(x), \mathcal{R}(y)\} = \left[Q^i(x) + Q^i(y)\right] \delta_{,i}(x,y), \tag{83}$$

where

$$Q^i = \eta^{ik}\mathcal{H}_k - uu^i\bar{\mathcal{H}} + (\gamma^{ik}u^2 + u^iu^k)\bar{\mathcal{H}}_k + 2u^j\gamma_{jk}\frac{\partial\tilde{U}}{\partial\gamma_{ik}}, \tag{84}$$

To fulfill equation (54) it is necessary to have

$$Q^i = \eta^{ik}(\mathcal{H}_k + \bar{\mathcal{H}}_k). \tag{85}$$

As equation (54) is to be valid for Dirac brackets, we can use second class constraints in the expression obtained for Q^i, i.e. replace $\bar{\mathcal{H}}$, $\bar{\mathcal{H}}_i$ by derivatives of the potential with respect for u, u^i, so this requires to treat equation (58) as a necessary condition for (54).

At last, let us check equation (56):

$$\begin{aligned}
\{\mathcal{R}_i(x), \mathcal{R}(y)\} &= \{\mathcal{H}_i(x), \mathcal{H}(y)\} + && (86)\\
&+ \{\bar{\mathcal{H}}_i(x), u\bar{\mathcal{H}}(y)\} + \{\bar{\mathcal{H}}_i(x), u^j\bar{\mathcal{H}}_j(y)\} + && (87)\\
&+ \{\mathcal{H}_i(x), \tilde{U}(y)\} + \{\bar{\mathcal{H}}_i(x), \tilde{U}(y)\} = && (88)\\
&= \mathcal{H}(x)\delta_{,i}(x,y) + && (89)\\
&+ u(y)\bar{\mathcal{H}}(x)\delta_{,i}(x,y) + u^j(y)(\bar{\mathcal{H}}_i(y) + \bar{\mathcal{H}}_j(x))\delta_{,i}(x,y) + && (90)\\
&+ 2\left(\eta_{im}(x)\frac{\partial\tilde{U}}{\partial\eta_{mn}}(x) + \gamma_{im}(x)\frac{\partial\tilde{U}}{\partial\gamma_{mn}}(x)\right)\delta_{,n}(x,y) + && (91)\\
&+ \left[\left(2\eta_{im}\frac{\partial\tilde{U}}{\partial\eta_{mn}} + 2\gamma_{im}\frac{\partial\tilde{U}}{\partial\gamma_{mn}}\right)_{,n} - \eta_{mn,i}\frac{\partial\tilde{U}}{\partial\eta_{mn}} - \gamma_{mn,i}\frac{\partial\tilde{U}}{\partial\gamma_{mn}}\right]\delta(x,y).
\end{aligned}$$

Proceeding in the same way to the previous calculation, and interpreting this bracket as Dirac bracket, we can use second class constraints in the expressions obtained. Then after exploiting formulas (82), it is possible to demonstrate that in order to fulfill equation (56) the potential has to satisfy condition (57).

B Second class constraints and Dirac brackets in bigravity

To make our presentation compact let us introduce notations: $u^a = (u, u^i)$, $\pi_a = (\pi_u, \pi_{u^i})$, $\bar{\mathcal{H}}_a = (\bar{\mathcal{H}}, \bar{\mathcal{H}}_i)$, $a = 1, .., 4$. We denote eight second class

constraints as $\chi_A, \ = 1,..,8$, so now we have

$$\chi_A = \left(\pi_a, \bar{\mathcal{H}}_a + \frac{\partial \tilde{U}}{\partial u^a}\right), \tag{92}$$

Then matrix of Poisson brackets for these constraints has the following structure

$$||\{\chi_A(x), \chi_B(y)\}|| = \left(\begin{array}{cc} \mathbf{0} & -\mathbf{L}(x)\delta(x,y) \\ \mathbf{L}(x)\delta(x,y) & \mathbf{K}(x,y) \end{array} \right), \tag{93}$$

where

$$\mathbf{L}_{ab}(x) = \frac{\partial^2 \tilde{U}}{\partial u^a \partial u^b}(x), \tag{94}$$

$$\mathbf{K}_{ab}(x,y) = \left\{ \bar{\mathcal{H}}_a(x) + \frac{\partial \tilde{U}}{\partial u^a}(x), \bar{\mathcal{H}}_b(y) + \frac{\partial \tilde{U}}{\partial u^b}(y) \right\}, \tag{95}$$

we suppose that in the general case matrix $\mathbf{L}$ is invertable, then matrix $||\{\chi_A, \chi_B\}||$ is invertable too, and its inverse is as follows

$$\mathbf{C} = \left(\begin{array}{cc} \mathbf{L}^{-1}(x)\mathbf{K}(x,y)\mathbf{L}^{-1}(y) & \mathbf{L}^{-1}(x)\delta(x,y) \\ -\mathbf{L}^{-1}(x)\delta(x,y) & \mathbf{0} \end{array} \right). \tag{96}$$

Dirac brackets are given in the following way

$$\{F, G\}_D = \{F, G\} - \int dx \int dy \{F, \chi_A(x)\} \mathbf{C}^{AB}(x,y) \{\chi_B(y), G\} \tag{97}$$

and in cases when both functionals F, G do not depend on variables u, u^i, π_u, π_{u^i}, these brackets coincide with the Poisson ones. Therefore, if constraints are explicitely solved for u, u^i, and these variables are replaced by constraint solutions, then for the rest of variables $\eta_{ij}, \Pi^{ij}, \ \gamma_{ij}, \pi^{ij}, \ \psi_A, \Pi^A, \ \phi_A, \pi^A$ Dirac brackets coinside with canonical Poisson brackets:

$$\{F, G\}_D = \int d^3x \left(\frac{\delta F}{\delta \eta_{ij}} \frac{\delta G}{\delta \Pi^{ij}} + \frac{\delta F}{\delta \psi_A} \frac{\delta G}{\delta \Pi^A} + \frac{\delta F}{\delta \gamma_{ij}} \frac{\delta G}{\delta \pi^{ij}} + \frac{\delta F}{\delta \phi_A} \frac{\delta G}{\delta \pi^A} - (F \leftrightarrow G) \right). \tag{98}$$

But sometimes an explicit solving of constraints may be difficult, then one may write bigravity Hamiltonian by means of potential $\tilde{U}$, which is a function of variables u, u^i and of the two induced metrics η_{ij}, γ_{ij} (66). In such a situation one has to use nontrivial Dirac brackets in obtaining Hamiltonian equations or in solving other problems:

$$\{\gamma_{mn}(x), u^a(y)\}_D = -\mathbf{L}^{-1ab}(y) \{\gamma_{mn}(x), \bar{\mathcal{H}}_b(y)\}, \tag{99}$$

$$\{\pi^{mn}(x), u^a(y)\}_D = \mathbf{L}^{-1\,ab}(y)\left(\frac{\partial^2\tilde{U}}{\partial u^b\partial\gamma_{mn}}\delta(x,y) - \{\pi^{mn}(x), \bar{\mathcal{H}}_b(y)\}\right), \quad (100)$$

$$\{\Pi^{mn}(x), u^a(y)\}_D = \mathbf{L}^{-1\,ab}\frac{\partial^2\tilde{U}}{\partial u^b\partial\eta_{mn}}\delta(x,y). \quad (101)$$